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Symmetry Breaking on a Model of Five-Mode Truncated Navier–Stokes Equations

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A symmetryless model of nonlinear first-order differential equations, obtained by perturbing a known model of five-mode truncated Navier–Stokes equations, is studied. Some interesting phenomena, such as the existence of an infinite sequence of bifurcations in a very narrow range of the parameter and the simultaneous presence of a strange attractor either with two fixed attracting points or with a periodic attracting orbit, are shown. Furthermore, two new sequences of period doubling bifurcations are found in the unperturbed model.

KEY WORDS: Symmetry breaking; Navier–Stokes equations; turbulence; strange attractor; period doubling bifurcation; hysteresis.

1. INTRODUCTION

In recent years much effort has been devoted to the attempt to give a mathematical interpretation to the phenomenon of turbulence in fluids. A relevant contribution to the progress in this line of research has certainly come from a large number of numerical studies on simple models of nonlinear evolution equations, which exhibit a transition to a stochastic behavior when one or more parameters go beyond certain critical values.

The Lorenz system,(1) consisting of three first-order ordinary differential equations representing a flow in three-dimensional space, and the Hénon mapping,(2) transforming the plane into itself, are surely the best known models. For a sufficiently detailed knowledge of the phenomenology of these models, several numerical studies were necessary. This emphasizes the importance of numerical work and at the same time testifies to the fact that any numerical investigation, no matter how accurate, can in general explain only part of the phenomenology of a model.

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A valid justification of the incompleteness of numerical investigations has to be found in the fact that the phenomenology of the model is often so varied and complicated that it cannot be completely defined. Two typical examples of this fact are represented just by the Lorenz and Hénon models.

Another factor must be considered: some phenomena, sometimes relevant, may occur in such narrow ranges of the parameter as to make all the investigations vain.

There are, however, other factors concurring to make numerical studies partial and incomplete: our experience is often too limited with respect to the many situations which may occur and the difficulty in adjusting effective and rigorous computing algorithms.

In this paper we intend to provide further support to the above considerations by producing the results related to the study of an interesting system of nonlinear differential equations. This system exhibits some phenomena which appear very significant, such as the existence of an infinite sequence of bifurcations in a parameter interval smaller than 0.001 and the simultaneous occurrence (hysteresis) of a strange attractor either with two stable fixed points or with a stable closed orbit.

The model we study here is obtained by perturbing a known model of five-mode truncated Navier–Stokes equations,[3,4] exhibiting a rather complicated phenomenology, which is, however, justified and simplified by the occurrence of a symmetry group. The role of these symmetries appears so significant that it is natural to wonder what happens if they are in some way broken. This can be done by adding two perturbative terms r_1 and r_5 to the first and the fifth equations, respectively.[2] So, if r_1 and r_5 are both different from zero, we obtain the following symmetryless system:

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5 + r_1 \\
\dot{x}_2 &= -9x_2 + 3x_1x_3 \\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + r \\
\dot{x}_4 &= -5x_4 - x_1x_5 \\
\dot{x}_5 &= -x_5 - 3x_1x_4 + r_5
\end{aligned} \tag{1}$$

If the unperturbed system ($r_1 = r_5 = 0$) is structurally stable, as it appears logical to suppose, the phenomenology of (1) must move with continuity and gradualness from that seen in Ref. 3 and 4, when r_1 and r_5 are slightly and continuously increased from zero. In other words, for each pair (r_1, r_5) we have a one-parameter family of differential equations

[2] This is the same as assuming an external force $\mathbf{f}$ (the periodic volume force on the fluid) also acting on the modes $\mathbf{k}_1$ and $\mathbf{k}_5$.

$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, r)$, whose behavior should be qualitatively analogous to that of the one-parameter families corresponding to "near" pairs $(\tilde{r}_1, \tilde{r}_5)$.

Our purpose here is not to study system (1) as a perturbation of the known model, i.e., with r_1 and r_5 both varying in a neighborhood of zero. We intend simply to study a symmetryless model obtained from (1) by assigning for r_1 and r_5 two values large enough to cause considerable changes in the phenomenology of the unperturbed system. As suggested by some preliminary experiments, we assume $r_1 = r_5 = 1$. However, we will refer in the following to this particular model as the perturbed one.

2. THE UNPERTURBED MODEL

Before we describe the numerical results of our study on the perturbed model, it seems useful to report a concise summary of the phenomenology of the unperturbed one, i.e., of the system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5 \\ \dot{x}_2 &= -9x_2 + 3x_1x_3 \\ \dot{x}_3 &= -5x_3 - 7x_1x_2 + r \\ \dot{x}_4 &= -5x_4 - x_1x_5 \\ \dot{x}_5 &= -x_5 - 3x_1x_4 \end{aligned} \tag{2}$$

In this way we hope to facilitate the understanding of the perturbed model which will be described later on. We hope also to provide the elements necessary to point out the differences in behavior between the two models. Therefore, we report from Refs. 3 and 4, as briefly as possible, the known results on model (2), with the addition of some new very recently found ones.[3]

(a) For $0 < r \leqslant R_1' = 5\sqrt{\frac{3}{2}}$ there is only one fixed point P_0, which is stable and globally attracting.

(b) For $R_1' < r \leqslant R_2' = \frac{80}{9}\sqrt{\frac{3}{2}}$ there are two other stable attractive fixed points P_α bifurcated at $r = R_1'$ from P_0, as it has become unstable (α is the sign of the coordinate x_1).

(c) For $R_2' < r \leqslant R_3' \simeq 22.854$[4] there are four more stable attracting stationary solutions $P_{\alpha\beta}$ (β represents the sign of the coordinate x_5). The points $P_{\alpha+}$ and $P_{\alpha-}$ bifurcate from P_α at $r = R_2'$ as P_α becomes unstable.

[3] To better connect the phenomenologies of the two models, it is convenient to introduce new notations for the different sequences of periodic orbits (with respect to the ones of Ref. 4).

[4] Most of the critical values R_i', $i \geqslant 3$, are taken from Ref. 4 rounded off at the third decimal figure.

(d) For $r = R_3'$ each point $P_{\alpha\beta}$ gives rise, via a direct Hopf bifurcation, to a closed orbit $\mathcal{K}_{\alpha\beta}^0$, which is stable and attractive. The four orbits $\mathcal{K}_{\alpha\beta}^0$ are symmetric (in virtue of the symmetries of the model) and then they have the same identical behavior as r increases.

Starting from the $\mathcal{K}_{\alpha\beta}^0$ four identical sequences $\mathcal{K}_{\alpha\beta}^i$ of symmetric periodic orbits take place in connection with an infinite sequence of bifurcations, which exhausts itself for $r = R_5' \simeq 28.669$. At the ith bifurcation, $i = 0, 1, \ldots,$ each orbit $\mathcal{K}_{\alpha\beta}^i$ becomes unstable because a real eigenvalue of the Liapunov matrix of the Poincaré map crosses the unit circle through -1, giving rise to the orbit $\mathcal{K}_{\alpha\beta}^{i+1}$, which is stable and attractive and has a doubled period.

(e) For $r = R_4' \simeq 28.663$ four more symmetric periodic orbits arise, which are stable and attracting. They have a spatial structure different from that of the $\mathcal{K}_{\alpha\beta}^i$. In fact, while each $\mathcal{K}_{\alpha\beta}^i$ winds up around the only fixed point $P_{\alpha\beta}$, each of these new orbits winds up around two points, making three loops around one point, then two around the other point, and so on. More precisely we have the orbit $\mathcal{Q}_+^0$, which goes twice around P_{-+} and three times around P_{++}, the orbit $\overline{\mathcal{Q}}_+^0$, which makes three loops around P_{-+} and two around P_{++}, the orbit $\mathcal{Q}_-^0$, with two loops around P_{--} and three around P_{+-}, and the orbit $\overline{\mathcal{Q}}_-^0$ making the symmetric of $\mathcal{Q}_-^0$. With regard to the four orbits $\mathcal{Q}_\beta^0$ and $\overline{\mathcal{Q}}_\beta^0$ it is useful to remark that two of them are entirely contained in the half-space $x_5 > 0$ and the other two in $x_5 < 0$: the sign β just refers to that half-space.

A second sequence of infinite period doubling bifurcations, accumulating at $r = R_8' \simeq 28.720$, gives rise to four more identical sequences of periodic orbits $\mathcal{Q}_+^i$, $\overline{\mathcal{Q}}_+^i$, $\mathcal{Q}_-^i$, $\overline{\mathcal{Q}}_-^i$, $i = 0, 1, \ldots,$ with the same characteristics of the sequences $\mathcal{K}_{\alpha\beta}^i$.

(f) For $28.7033 \simeq R_6' \leqslant r < R_7' \simeq 28.7068$, simultaneously to the four orbits $\mathcal{Q}^1$ (i.e., $\mathcal{Q}_+^1$, $\overline{\mathcal{Q}}_+^1$, $\mathcal{Q}_-^1$, $\overline{\mathcal{Q}}_-^1$) four extra sequences of symmetric orbits $\mathcal{B}_+^i$, $\overline{\mathcal{B}}_+^i$, $\mathcal{B}_-^i$, $\overline{\mathcal{B}}_-^i$, $i = 0, 1, \ldots,$ are present. Each of them has its own stability range but a very small basin of attraction. On the spatial characteristics of the $\mathcal{B}$'s, the same considerations made for the $\mathcal{Q}$'s hold with one difference: each orbit $\mathcal{B}$ makes three loops around a point and then only one around the other point (Fig. 1).

(g) For $R_8' \leqslant r < R_9' \simeq 30.189$ two symmetric strange attractors are present, one located in $x_5 > 0$ and the other in $x_5 < 0$. Every randomly chosen initial point tends to describe stochastic trajectories on one of these two attractors.

(h) At $r = R_9'$ four new symmetric attractive closed orbits $\mathcal{C}_+^0$, $\overline{\mathcal{C}}_+^0$, $\mathcal{C}_-^0$, $\overline{\mathcal{C}}_-^0$ appear. They have spatial properties completely analogous to those of the orbits $\mathcal{Q}$ and $\mathcal{B}$, winding up twice around one point and then once around the other point (Fig. 2). Also the orbits $\mathcal{C}^0$ originate four identical

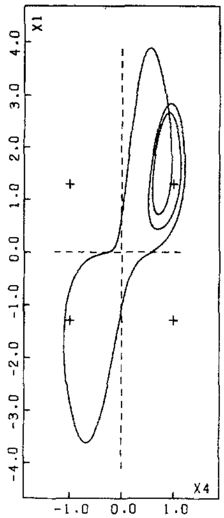

Fig. 1. Projection on the plane (x_4, x_1) of an orbit $\mathfrak{B}$ (the orbit $\mathfrak{B}^0_-$) for $r = 28.704$. The crosses indicate the fixed points $P_{\alpha\beta}$.

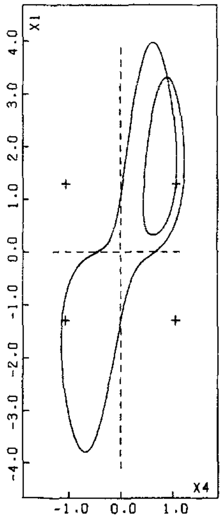

Fig. 2. Projection on the plane (x_4, x_1) of an orbit $\mathcal{C}$ (the orbit $\mathcal{C}^0_-$) for $r = 30.30$.

sequences $\mathcal{C}^i$, because of another sequence of infinite period doubling bifurcations, accumulating at $r = R'_{10} \simeq 30.543$.

(i) For $r = R'_{10}$ the two strange attractors, which disappeared at $r = R'_9$ owing to the arising of the stable $\mathcal{C}^0$'s, reappear. So, for $R'_{10} \leqslant r < R_{11} \simeq 33.439$, the motion is stochastic again.

(j) For $r \geqslant R'_{11}$ any trajectory rapidly becomes periodic because of two stable attracting closed orbits Γ_β (β has the above meaning).

The whole phenomenology is graphically summarized in Fig. 3. A brief glance at the picture is sufficient to show how the behavior of the unperturbed model is complicated, even if simplified by the occurrence of symmetries.

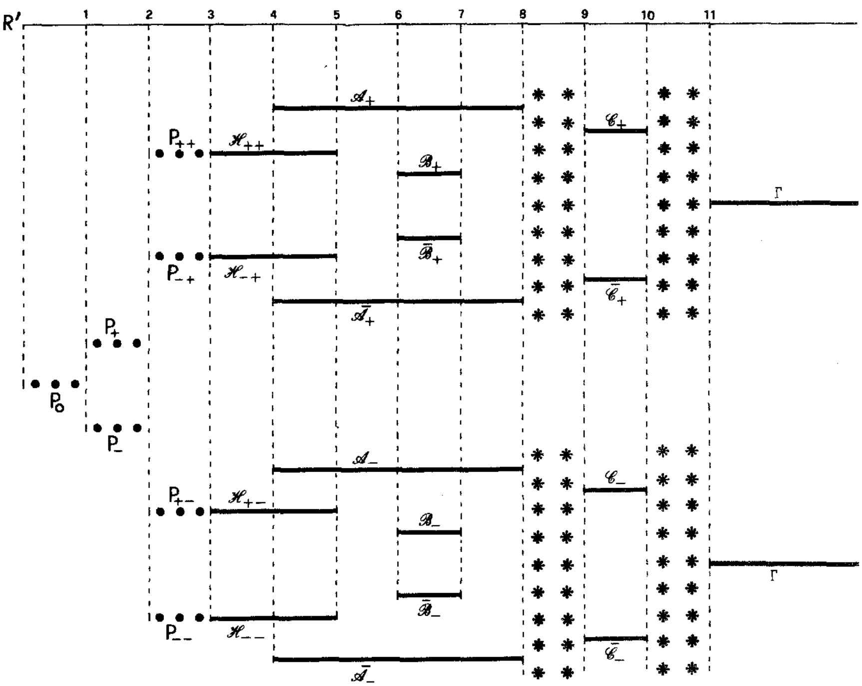

Fig. 3. Graphical summary of the phenomenology exhibited by the unperturbed model as r varies. A sequence of spots is used to represent a stable fixed point, a continuous thick line is used for a stable periodic orbit, a set of stars for turbulent regime. The critical points R'_i, here indicated by their subscript and represented as equispaced, have the following values: $R'_1 = 5\sqrt{\frac{3}{2}}$, $R'_2 = \frac{80}{9}\sqrt{\frac{3}{2}}$, $R'_3 \simeq 22.854$, $R'_4 \simeq 28.663$, $R'_5 \simeq 28.669$, $R'_6 \simeq 28.7033$, $R'_7 \simeq 28.7068$, $R'_8 \simeq 28.720$, $R'_9 \simeq 30.189$, $R'_{10} \simeq 30.543$, $R'_{11} \simeq 33.439$.

Table I. Bifurcation Points of the Sequences $\mathcal{B}^i$ and $\mathcal{C}^i$

	$\mathcal{B}^i$	$\mathcal{C}^i$
ρ_0	28.7033	30.189
ρ_1	28.7055	30.406
ρ_2	28.70652	30.5191
ρ_3	28.70674	30.5378
ρ_4	—	30.5418
⋮	⋮	⋮
ρ_∞	28.7068	30.543

Finally, we focus our attention a little on the orbits $\mathcal{B}$ and $\mathcal{C}$. They had not been found in Refs. 3 and 4. For this reason it appears not useless to complete the phenomenology of the unperturbed model by computing with good accuracy some of the first bifurcation points ρ_i of the two sequences, ρ_i being the value of r for which the ith orbit arises. In Table I we list the values ρ_i we have calculated, also including a rough approximation ρ_∞ for the accumulation point of each sequence. The large enough period of the orbits $\mathcal{B}^0$ and $\mathcal{C}^0$ and the consequent elevated costs necessary to obtain accurate results, have considerably limited our computations.

The fact that the $\mathcal{B}$'s and $\mathcal{C}$'s had not been previously found, can be easily explained. As regards the orbits $\mathcal{B}$, their very narrow "life" r interval and their very small basin of attraction (in connection to the quite large one of the coexistent orbits $\mathcal{A}^1$) make them impossible to be found. In the following we will see how, just through the perturbations adduced on the model, range of existence and basin of attraction can be considerably enlarged in such a way that the orbits can be found. Differently from the $\mathcal{B}$'s, the orbits $\mathcal{C}$ are present in a sufficiently large r interval and attract any randomly chosen initial point. Therefore, the reason why they had not been found in Refs. 3 and 4 is very simple: no solution of (2) was studied for r belonging to $[R_9', R_{10}']$.

3. THE PERTURBED MODEL

Consider the system of equations

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5 + 1\\
\dot{x}_2 &= -9x_2 + 3x_1x_3\\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + r\\
\dot{x}_4 &= -5x_4 - x_1x_5\\
\dot{x}_5 &= -x_5 - 3x_1x_4 + 1
\end{aligned} \tag{3}$$

A preliminary consideration simplifies the exposition of the phenomenology of this model. Each attractor of (3) corresponds to an analogous one exhibited by the unperturbed model, as was to be expected by the considerations made at the end of the Introduction. Such a fact is very useful because it allows us to use the same notation for describing the behavior of the two models, which thus become immediately comparable.

Fixed Points

The fixed points of system (3) have coordinates

$$\left(x_1, \frac{rx_1}{7x_1^2+15}, \frac{3r}{7x_1^2+15}, \frac{-x_1}{5-3x_1^2}, \frac{5}{5-3x_1^2}\right)$$

where x_1 satisfies the following equation of the ninth degree:

$$(2x_1-1)(7x_1^2+15)^2(5-3x_1^2)^2-12rx_1(5-3x_1^2)^2+20x_1(7x_1^2+15)^2=0 \tag{4}$$

A numerical study of the real solutions of equation (4), each of them yielding a fixed point for the system (3), gives the following results:

(i) For $0<r<R_1\simeq 10.8661$ there is only one real solution, corresponding to the fixed point P_{++}.

(ii) For $R_1 \leqslant r < R_2 \simeq 16.6024$ there are three real solutions for equation (4): the fixed points P_0 and P_{-+} are added to P_{++}.

(iii) For $R_2 \leqslant r < R_5 \simeq 20.2104$ the stationary solutions are five: the three previous ones plus P_+ and P_{+-}.

(iv) For $r \geqslant R_5$ the system (3) has seven fixed points: in addition to $P_{++}, P_0, P_{-+}, P_+, P_{+-}$, we have P_- and P_{--}.

It is easy to verify that, whereas P_0, P_+, and P_- are always unstable, P_{++}, P_{-+}, P_{+-}, and P_{--} are stable and attractive up to $r=R_4\simeq$ 19.7996, $R_3\simeq 19.4636$, $R_{13}\simeq 25.9272$, $R_{14}\simeq 26.5689$, respectively. At each of these four critical points a direct Hopf bifurcation takes place. In consequence of such a bifurcation a stable attracting closed orbit $\mathcal{K}^0_{\alpha\beta}$ comes out of each point $P_{\alpha\beta}$.

An interesting remark: whereas the point P_{++} exists for every r, the other six ones arise in pairs, each pair having its own origin and consisting of a stable point and an unstable one. This way of arising, which is different from that of the fixed point in the unperturbed model, is clearly due to the absence of symmetries in the system (3).

Table II. Bifurcation Points of the Sequences $\mathcal{H}^i_{\alpha\beta}$

	$\mathcal{H}^i_{++}$	$\mathcal{H}^i_{-+}$	$\mathcal{H}^i_{+-}$	$\mathcal{H}^i_{--}$
ρ_0	19.7995	19.4636	25.9272	26.5688
ρ_1	23.270	22.273	33.56	34.42
ρ_2	23.388	22.380	33.866	34.742
ρ_3	23.404	22.394	33.899	34.774
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ρ_∞	23.409	22.398	33.908	34.782

Sequences of Infinite Periodic Orbits

Each orbit $\mathcal{H}^0_{\alpha\beta}$ which originates from the point $P_{\alpha\beta}$ becomes unstable with an eigenvalue -1 bifurcating into a new stable periodic orbit $\mathcal{H}^1_{\alpha\beta}$ with double period. The phenomenon of period doubling bifurcations continues also with $\mathcal{H}^1_{\alpha\beta}$, giving rise in this manner to four different sequences of orbits $\mathcal{H}^i_{\alpha\beta}$. Each sequence takes place in its own r interval and with its own sequence of bifurcation points $\{\rho_i\}$, $i = 0, 1, \ldots$, ρ_i being the critical value of r for which the orbit $\mathcal{H}^i$ arises.

Table II shows the values ρ_i for the four sequences $\mathcal{H}^i_{\alpha\beta}$, by including only the first three ones for what concerns the period doubling bifurcations,[5] and, in addition, a rough approximation ρ_∞ for the value of r for which the sequence exhausts itself. Such an approximation [we suggest keeping in mind Feigenbaum's theory (see Ref. 4)] is useful because it permits for each sequence $\mathcal{H}^i_{\alpha\beta}$ a definition of its "life" interval. In fact, by reading from Table II, we can assume $R_6 \simeq 22.398$, $R_9 \simeq 23.409$, $R_{15} \simeq 33.908$, $R_{19} \simeq 34.782$ and then have the interval (R_4, R_9) for the sequence $\mathcal{H}^i_{++}$, (R_3, R_6) for $\mathcal{H}^i_{-+}$, (R_{13}, R_{15}) for $\mathcal{H}^i_{+-}$, (R_{14}, R_{19}) for $\mathcal{H}^i_{--}$.

As is evident from Table II, the four sequences $\mathcal{H}^i_{\alpha\beta}$ describe here four different although similar stories. We recall that in the unperturbed model the four stories are perfectly identical.

Now turn our attention to the orbits $\mathcal{A}^i$, $\mathcal{B}^i$, $\mathcal{C}^i$. If one studies the asymptotic behavior of the solutions of system (3) as r varies and with several random initial data, it is easy to observe the presence of three different sequences of attracting periodic orbits. These can be immediately identified, thanks to their spatial structure, with the sequences $\mathcal{C}^i_+$, $\mathcal{B}^i_-$, $\mathcal{C}^i_-$.

[5] Since the phenomenon of sequences of infinite bifurcations has already been studied in great detail in Ref. 4, it seems sufficient to consider here only three period doubling bifurcations. Also concerning the critical points ρ_i, $i = 1, 2, 3$, it appears useless to compute them with higher precision.

By recalling that in the unperturbed model four sequences $\mathcal{A}^i$, four $\mathcal{B}^i$'s, and four $\mathcal{C}^i$'s are present, the question arises whether the remaining nine are still present or not. To answer this question we have adopted different search techniques. Taking the orbits of model (2) as a starting point, we have tried to follow them when the parameters r_1 and r_5 are slightly and continuously increased from zero, either simultaneously or separately. There was the hope of being able to follow the orbits up to $r_1 = r_5 = 1$.

Our attempt has met with success as far as the sequences $\mathcal{B}^i_+$, $\mathcal{A}^i_-$, and $\bar{\mathcal{A}}^i_-$ are concerned. To give an idea of the difficulties in obtaining such a result, it is sufficient to say that the orbit $\bar{\mathcal{A}}^0_-$ is now present with a very small stability range: about 0.0006. In the unperturbed model this range is about 0.03.

On the contrary, all our efforts to find the sequences $\mathcal{A}^i_+, \bar{\mathcal{A}}^i_+, \bar{\mathcal{B}}^i_+, \bar{\mathcal{B}}^i_-$, $\bar{\mathcal{C}}^i_+, \bar{\mathcal{C}}^i_-$ have proved useless. They are stable in an interval which rapidly becomes smaller and sensibly shifts with respect to r. Very soon it becomes impossible to follow the orbits as r_1 and r_5 increase. For this reason we cannot establish whether these sequences are still present in a very small range or not any more.

Table III collects, analogously to Table II, some numerical data relative to the sequences we have found. Also in this case it is useful to define their "life" interval through the critical values ρ_0 and ρ_∞. Letting $R_7 \simeq 22.972$, $R_8 \simeq 23.006$, $R_{10} \simeq 23.488$, $R_{11} \simeq 23.867$, $R_{16} \simeq 34.411$, $R_{17} \simeq 34.5892$, $R_{18} \simeq 34.5900$, $R_{20} \simeq 34.849$, $R_{21} \simeq 35.497$, $R_{22} \simeq 35.596$, $R_{23} \simeq 37.571$, $R_{24} \simeq 41.022$, we have the following ranges: (R_7, R_8) for the sequence $\mathcal{B}^i_+$, (R_{10}, R_{11}) for $\mathcal{C}^i_+$, (R_{16}, R_{20}) for $\mathcal{B}^i_-$, (R_{17}, R_{18}) for $\bar{\mathcal{A}}^i_-$, (R_{21}, R_{22}) for $\mathcal{A}^i_-$, (R_{23}, R_{24}) for $\mathcal{C}^i_-$.

For precision's sake a remark must be made concerning the orbits $\mathcal{B}$. Their presence in the unperturbed model was unknown before studying system (3) and finding a periodic orbit (the orbit $\mathcal{B}^0_-$) with spatial characteristics different from those of the known orbits $\mathcal{A}$ and $\mathcal{C}$. Following this

Table III. Bifurcation Points of the Sequences $\mathcal{B}^i_+$, $\mathcal{C}^i_+$, $\mathcal{B}^i_-$, $\bar{\mathcal{A}}^i_-$, $\mathcal{A}^i_-$, $\mathcal{C}^i_-$

	$\mathcal{B}^i_+$	$\mathcal{C}^i_+$	$\mathcal{B}^i_-$	$\bar{\mathcal{A}}^i_-$	$\mathcal{A}^i_-$	$\mathcal{C}^i_-$
ρ_0	22.972	23.488	34.411	34.5892	35.497	37.571
ρ_1	22.989	23.835	34.69	34.5898	35.571	40.924
ρ_2	23.002	23.862	34.824	—	35.591	41.004
ρ_3	—	23.866	34.843	—	35.595	41.018
⋮	⋮	⋮	⋮	⋮	⋮	⋮
ρ_∞	23.006	23.867	34.849	34.5900	35.596	41.022

new orbit as the parameters r_1 and r_5 both decrease from one up to zero, we have found the orbits $\mathfrak{B}$ also in the unperturbed model.

Behavior at Intermediate and High r Regimes

Exactly as in model (2), if r is sufficiently high, any randomly chosen initial point tends to a periodic motion. This is due to the presence of two stable attracting closed orbits Γ_+ (in $x_5 > 0$) and Γ_- (in $x_5 < 0$) existing for $r \geqslant R_{12} \simeq 24.57$ and $r \geqslant R_{25} \simeq 49.99$, respectively. Such orbits, whose existence is easily verifiable, remain very likely also for r tending to infinity. We verified their existence up to $r = 500$, when they appear very enlarged and with a period much smaller than the initial one.

Consider now the intervals (R_9, R_{10}), (R_{11}, R_{12}), (R_{20}, R_{21}), (R_{22}, R_{23}) and (R_{24}, R_{25}). For values of r belonging to one of these intervals, there are solutions which tend asymptotically to an attractor having all the characteristics of "strange attractor." In fact the trajectories described by such solutions appear completely random, are condensed in a well-circumscribed region of the space, and sensitively depend on initial condition.

The behavior in the intervals (R_9, R_{12}) and (R_{20}, R_{25}) essentially repeats that of the unperturbed model in the interval (R_8', R_{11}'). The strange attractor, approached through a cascade of period doubling bifurcations, disappears owing to the arising of a stable closed orbit on the same invariant manifold containing it. A new sequence of period doubling bifurcations reinstates the strange attractor on the manifold. It definitively disappears when a stable periodic orbit Γ appears. A quite analogous phenomenology is exhibited by the Lorenz model too.[5]

A remarkable fact is represented by the phenomena of hysteresis still taking place in the five intervals considered above. In the two first ones we have the simultaneous occurrence of a strange attractor, located in the half-space $x_5 > 0$, with the two stable fixed points P_{+-} and P_{--}, located in $x_5 < 0$. In the last three intervals the strange attractor, which is in $x_5 < 0$, exists contemporarily to the stable orbit Γ_+, localized in $x_5 > 0$. It is easy to verify that, if we take several random initial points, no matter how r is chosen belonging to one of the five intervals, some solutions tend to turbulent motion, the others to laminar one.

4. CONCLUSION

In this paper we have presented the numerical results of our study on a system of nonlinear differential equations. Such a system is obtained by perturbing a known model of five-truncated Navier–Stokes equations in order to break all its symmetries. The perturbed model substantially repeats

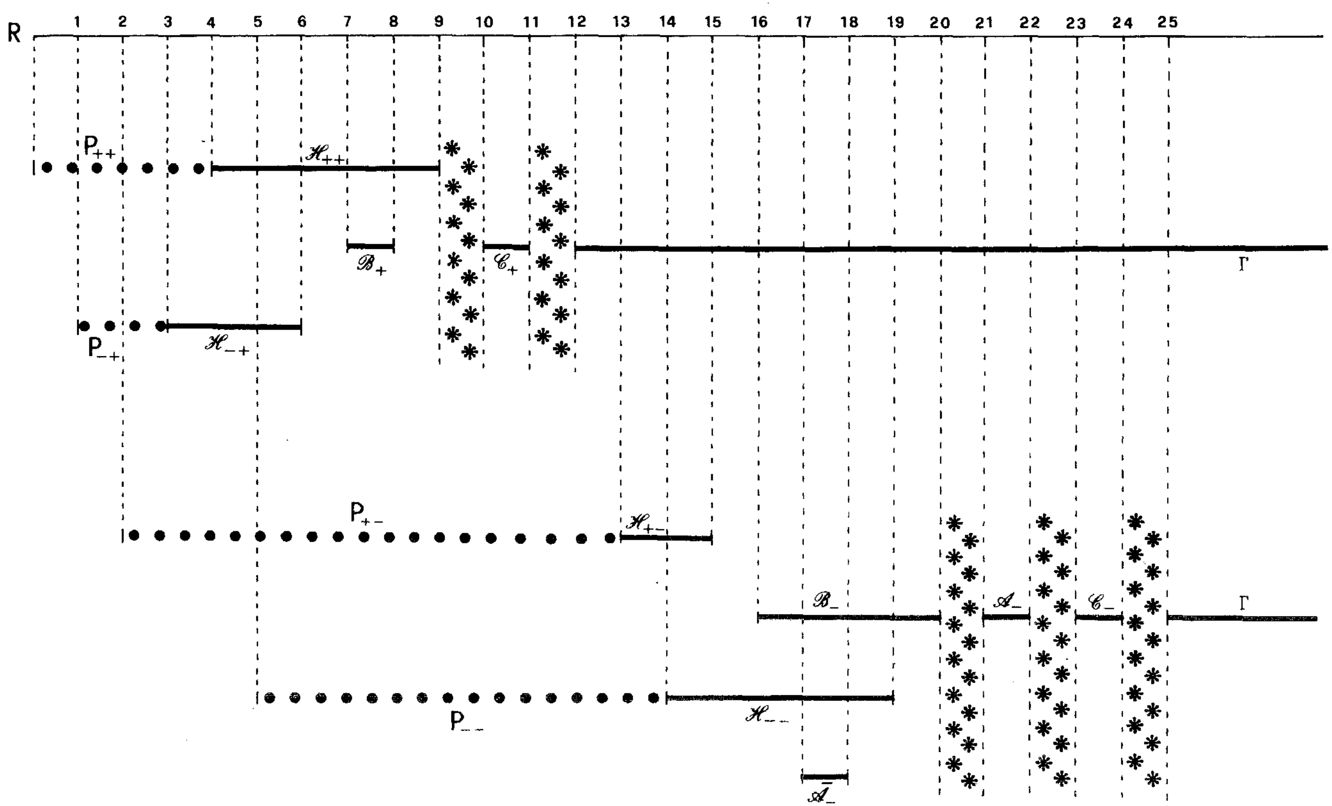

Fig. 4. Graphical summary (analogous to Fig. 3) of the phenomenology exhibited by the perturbed model as r varies. Critical values: $R_1 \simeq 10.866$, $R_2 \simeq 16.602$, $R_3 \simeq 19.464$, $R_4 \simeq 19.800$, $R_5 \simeq 20.210$, $R_6 \simeq 22.398$, $R_7 \simeq 22.972$, $R_8 \simeq 23.006$, $R_9 \simeq 23.409$, $R_{10} \simeq 23.488$, $R_{11} \simeq 23.867$, $R_{12} \simeq 24.57$, $R_{13} \simeq 25.927$, $R_{14} \simeq 26.569$, $R_{15} \simeq 33.908$, $R_{16} \simeq 34.411$, $R_{17} \simeq 34.5892$, $R_{18} \simeq 34.5900$, $R_{19} \simeq 34.782$, $R_{20} \simeq 34.849$, $R_{21} \simeq 35.497$, $R_{22} \simeq 35.596$, $R_{23} \simeq 37.571$, $R_{24} \simeq 41.022$, $R_{25} \simeq 49.99$.

all the most significant behaviors of the unperturbed one. Moreover, it presents some new phenomena, which enrich its already varied phenomenology.

Some remarks on the effects of symmetry breaking appear interesting.

First of all the occurrence of the symmetries caused, after the arising of the four symmetric points $P_{\alpha\beta}$ at $r = R_2'$, the origin of four perfectly identical stories. Then they were reduced to two as the two strange attractors appeared. The breaking of all the symmetries causes the four stories, although similar, to differ.

A second consideration concerns the phenomenology in more general terms: in the perturbed model it develops in a different way in the two half-spaces $x_5 > 0$ and $x_5 < 0$. As can be immediately verified looking at the summary of Fig. 4, the phenomenology in $x_5 > 0$ undergoes a contraction (with respect to the parameter r, obviously), whereas in $x_5 < 0$ there is an evident dilatation. Owing to this, special cases of hysteresis take place. In fact in the intervals (R_9, R_{10}) and (R_{12}, R_{13}) a strange attractor and two stable fixed points are present at the same time, while in (R_{20}, R_{21}), (R_{22}, R_{23}), and (R_{24}, R_{25}) there are a strange attractor and an attractive closed orbit. This fact, meaning that for the same value of the parameter a solution may tend either to stochastic motion or to laminar one depending on initial conditions, appears to be very interesting. As far as we are concerned, it is the first time that such a phenomenon is met in studying dissipative systems of differential equations, even if it is known to be present in the Hénon model.[6]

Finally, we consider the two facts which, in our opinion, represent the more interesting effects of symmetry breaking. The sequence of periodic orbits $\bar{\mathfrak{A}}^i_-$, that in the unperturbed model develops in a r interval a little larger than 0.05, in the perturbed one has an existence range less than 0.001. On the contrary, the sequence $\mathfrak{B}^i_-$ changes from a "life" interval less than 0.004 to one larger than 0.43. Therefore, while in the former case the perturbations cause a contraction by a factor smaller than $1/50$, in the latter one the effect is a dilatation by a factor larger than 100! By this two considerations immediately follow: On the one hand we find a confirmation of the fact that some attractors, in particular stable periodic orbits, sometimes even important, can take place in such small intervals as to be "invisible." On the other hand, a possible way to discover such attractors is suggested: through suitable perturbations their range of existence and basin of attraction can be enlarged as much as necessary to find them.

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REFERENCES

1. E. N. Lorenz, *J. Atmos. Sci.* **20**:130 (1963).
2. M. Hénon, *Commun. Math. Phys.* **50**:69 (1976).
3. C. Boldrighini, and V. Franceschini, *Commun. Math. Phys.* **64**:159 (1979).
4. V. Franceschini and C. Tebaldi, *J. Stat. Phys.* **21**:707 (1979).
5. V. Franceschini, *J. Stat. Phys.* **22**:397 (1980).
6. J. H. Curry, *Commun. Math. Phys.* **68**:129 (1979).